COOLING OF A SEMIINFINITE BODY WITH NONLINEAR HEAT LIBERATION FROM THE SURFACE

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Expressions are obtained which describe the changes in surface temperature of a semiinfinite body over short and long terms in problems with nonlinear boundary conditions.

We will present a technique for solution of problems with nonlinear boundary conditions using the following example, which is of practical interest:

$$\left(\frac{\partial}{\partial t}-a\frac{\partial^2}{\partial x^2}\right)T=0,\ 0\leqslant x<\infty,\ 0<t<\infty, \tag{1}$$

$$T|_{t=0}=T_0=\text{const}, \tag{2}$$

$$T|_{x=\infty}=T_0=\text{const}, \tag{3}$$

$$\lambda\frac{\partial T}{\partial x}\Big|_{x=0}=\sigma T_s^4. \tag{4}$$

We must find the surface temperature as a function of time $T = T_s(t)$.

The problem of Eqs. (1)-(4) was considered previously in [1]. The proposed solution technique is significantly simpler if it is not required that the temperature field be determined, and, moreover, it permits deriving the asymptotic cooling law for large time periods.

It is well known that Eqs. (1)-(3) lead to an expression relating T_s and $\partial T/\partial x|_{x=0}$ in the form (see, e.g., [2, 3])

$$-\sqrt{a}\,\frac{\partial(T_0-T)}{\partial x}\Big|_{x=0}=D^{1/2}(T_0-T_s)=\frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t\frac{T_0-T_s(\tau)}{\sqrt{t-\tau}}d\tau$$

or

$$-\sqrt{a}\,\frac{\partial T}{\partial x}\Big|_{x=0}=\frac{1}{\sqrt{\pi}}\int_0^t\frac{dT_s(\tau)}{d\tau}\frac{1}{\sqrt{t-\tau}}d\tau. \tag{5}$$

Eliminating $\partial T/\partial x|_{x=0}$ from Eqs. (5) and (4), we obtain an equation defining the desired function $T_s(t)$:

$$D^{1/2}(T_0-T_s)=-\frac{1}{\sqrt{\pi}}\int_0^t\frac{dT_s(\tau)}{d\tau}\frac{1}{\sqrt{t-\tau}}d\tau=\alpha T_s^4,\ \alpha=\sigma\sqrt{a}/\lambda. \tag{6}$$

We can find a solution of (6) in the form of a power series

$$T=T_0-\sum_{n=1}^{\infty}a_nt^{n/2}, \tag{7}$$

since the transform D^{ν} transforms a power series into another power series:

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$$D^{\nu} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)} t^{\mu-\nu}. \tag{8}$$

Substituting (7) in (6), using (8), and equating coefficients of similar powers of t, we obtain a system of algebraic equations for sequential determination of a_n, which then permits finding a solution in the form

$$T_s/T_0 = 1-(2/\sqrt{\pi})(\alpha T_0^3)\, t^{1/2} + 4\,(\alpha T_0^3)^2 t - [(4/3\sqrt{\pi}) + (2/\pi^{3/2})]\times \\ \times(\alpha T_0^3)^3 t^{3/2} + [(275/4\pi) + (48/\pi^{3/2}) - (12/\pi^2)](\alpha T_0^3)^4 t^2 - \ldots \tag{9}$$

We will study the convergence of series (9). Applying to Eq. (6) the operator $D^{-1/2}$, we obtain the equation

$$T_s = T_0 - \alpha D^{-1/2} T_s^4 = T_0 - \frac{\alpha}{\sqrt{\pi}} \int\limits_0^t \frac{T_s^4(\tau)}{\sqrt{t-\tau}}\, d\tau, \tag{10}$$

which can be used for the solution in place of (6).

We add to Eq. (10) the additional expression

$$T_s = T_0 + \frac{2\alpha}{\sqrt{\pi}} \sqrt{t}\, T_s^4 \tag{11}$$

and seek a solution in the form of (7), where we take $a_n \to b_n$. Using Eq. (8), it can be shown that $|b_n| \geqslant |a_n|$. The roots of the fourth-order algebraic equation are representable by a finite combination of radicals, so that the solution of Eq. (11) can be represented as a series in powers of $t^{1/2}$ having a finite radius of convergence. Then, in view of the inequality established above, series (9) has at least a finite radius of convergence. Equation (9) is convenient for practical calculations if $\alpha T_0^3 t^{1/2} \ll 1$.

To find the solution as $t \to \infty$, we consider the problem of calculating the asymptotic value of the integral

$$J = \int\limits_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\, d\tau. \tag{12}$$

We rewrite (12) in the form

$$J = \frac{1}{\sqrt{t}} \int\limits_0^t f(\tau)\, d\tau + \int\limits_0^t f(\tau) \left(\frac{1}{\sqrt{t-\tau}} - \frac{1}{\sqrt{\tau}}\right) d\tau = \frac{1}{\sqrt{t}} \int\limits_0^t f(\tau)\, d\tau + \sqrt{t} \int\limits_0^1 f(tz) \left[\frac{1}{\sqrt{1-z}} - 1\right] dz. \tag{13}$$

We assume that the function $f(t) \not\equiv 0$ and has the following properties: $f \geqslant 0$; $\int\limits_0^t f(\tau)\, d\tau < \infty$; $f(\tau) <$ const $t^{-\nu}$, where $1 < \nu < 2$. As $t \to \infty$, the first term of (13) gives

$$J \approx \frac{1}{\sqrt{t}} \int\limits_0^t f(\tau)\, d\tau \approx \frac{1}{\sqrt{t}} \int\limits_0^{\infty} f(\tau)\, d\tau. \tag{14}$$

The second term of (13) for f(t) of the form indicated is greater than the function

$$\frac{\text{const}}{t^{\nu-\frac{1}{2}}} \int\limits_0^1 z^{-\nu} \left[\frac{1}{\sqrt{1-z}} - 1\right] dz < \frac{\text{const}}{t^{\nu-\frac{1}{2}}},$$

which falls off as $t \to \infty$ more rapidly than $t^{-1/2}$ for the ν values indicated. Thus, the asymptotic expression of (12) as $t \to \infty$ is given by (14). Using the latter, we obtain from (6)

$$\frac{1}{\sqrt{\pi t}}(T_0 - T_s) \approx \alpha T_s^4 .$$

Neglecting the small quantity T_s within the parentheses, we find the final expression for the change in surface temperature at large times:

$$T_s \approx \left(\frac{T_0}{\alpha\sqrt{\pi}}\right)^{1/4} t^{-1/8}. \tag{15}$$

The expression obtained corresponds to the function $f(t) = t^{-5/4}$ in the second term of (13), for which (14) is valid. Similarly, if instead of (4) the heat liberation law has the form

$$\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = f(T_s) > 0, \tag{16}$$

one can obtain a solution for short times in the form of (7). If $f(T_s)$ is a polynomial of fourth degree in T_s (without a free term), its convergence is proved just as in the derivation of (9).

For large times, if $f(T_s) = \sigma T_s^{\mu} + O(T_s^{\mu+\varepsilon})$, $\varepsilon > 0$, we have

$$T_s \approx \left(\frac{T_0}{\alpha\sqrt{\pi}}\right)^{\frac{1}{\mu}} t^{-\frac{1}{2\mu}}. \tag{17}$$

In accordance with the conditions for deriving (14), Eq. (17) is proven if $1 < (1/2\mu) + 1 < 2$, i.e., $\mu > 1/2$. The question of the validity of (17) for $\mu \leqslant 1/2$ has not been studied. The case $\mu = 1$, where the problem reduces to the linear case, can be checked by conventional methods.

NOTATION

a, thermal diffusivity; a_n, b_n, series coefficients; D^{γ}, partial differentiation symbol; f, arbitrary function; J, integral; T, temperature; t, time; x, coordinate; α, combination constant; ε, constant; λ, thermal conductivity; σ, nonlinear radiation law constant; τ, auxiliary integration variable; μ, ν, exponents. Subscripts: s, surface; 0, initial moment.

LITERATURE CITED

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